RP Current Trends In Applied Sciences (e-ISSN: 2583-7486)
Vol. 4, No. 4 (October – December 2025) pp. 98–115

Cite this article: A. Kumar, Modulational amplification via nonlinear plasmon– acoustical phonon interaction in weakly piezoelectric magnetoactive doped III–V semiconductors, *RP Cur. Tr. Appl. Sci*. **4** (2025) 98–115.

Original Research Article

Modulational amplification via nonlinear plasmon– acoustical phonon interaction in weakly piezoelectric magnetoactive doped III–V semiconductors

Arun Kumar*

Assistant Professor, Department of Physics, Government College Bahadurgarh (Jhajjar), 124507, Haryana, India
*Corresponding author, E-mail: arun250981@gmail.com

ARTICLE HISTORY
Received: 12 Sept. 2025
Revised: 4 Dec. 2025
Accepted: 6 Dec. 2025
Published: 10 Dec. 2025

KEYWORDS
Modulational amplification; plasmon–acoustic phonon interaction; threshold pump amplitude; growth rate; magnetoactive III–V semiconductors.

ABSTRACT
The present investigation provides a comprehensive theoretical analysis of modulational amplification resulting from the nonlinear interaction between plasmons and acoustical phonons in weakly piezoelectric, magnetically biased, heavily doped III–V semiconductor materials. The nonlinear coupling mechanism is attributed to the third-order optical susceptibility, which originates from the nonlinear component of the induced current density generated within the semiconductor under the influence of an intense electromagnetic field. The analytical framework has been formulated using the coupled-mode theory, which effectively describes the energy exchange between interacting plasma, phonon, and optical waves in a nonlinear medium. Based on this theoretical formulation, explicit analytical expressions have been derived for two important physical parameters governing the modulational instability process: (i) the threshold pump-field amplitude required to initiate modulational amplification, and (ii) the growth rate of the generated modulation sidebands. These expressions reveal the conditions under which the nonlinear interaction becomes sufficiently strong to overcome dissipative effects, thereby enabling efficient amplification of the modulated wave.

1. Introduction

The remarkable advancement of nonlinear optics has been primarily driven by the invention and continuous improvement of highly coherent and high-intensity radiation sources, particularly lasers. The availability of intense coherent electromagnetic waves has enabled the observation and exploitation of a wide variety of nonlinear optical phenomena that were previously inaccessible under weak-field conditions. Consequently, extensive progress has been achieved not only in the theoretical understanding of nonlinear light–matter interactions but also in the investigation of nonlinear processes occurring in solid-state materials. These developments have significantly expanded the scope of optoelectronics, leading to the realization of numerous advanced photonic technologies and high-performance optical devices.

Among the diverse nonlinear phenomena exhibited by solid-state media, modulational interactions have attracted considerable scientific and technological interest because of their fundamental role in nonlinear wave propagation, energy transfer mechanisms, and optical signal amplification. Modulational processes arise from the interplay between material nonlinearity and wave dispersion, resulting in the amplification of perturbations imposed on an optical carrier wave. Owing to their capability to efficiently manipulate the amplitude, phase, and frequency characteristics of optical signals, modulational interactions have become indispensable in modern optoelectronic engineering. These nonlinear mechanisms provide the physical basis for the development of compact, high-speed, and energy-efficient photonic components that are extensively employed in advanced optical communication and information-processing systems [1–7].

One of the most significant applications of modulational interactions is found in the design and operation of optical modulators, which constitute essential building blocks of contemporary photonic technologies. Optical modulators exploit nonlinear optical effects to dynamically control the properties of propagating electromagnetic waves, thereby enabling efficient encoding and transmission of information. Such devices play a crucial role in the fabrication of high-speed photonic switches, integrated optical interconnects, wavelength-selective communication networks, and tunable coherent transmitters and receivers. Furthermore, nonlinear optical modulators offer several advantages, including low insertion loss, enhanced modulation efficiency, high switching speed, broad operational bandwidth, and compatibility with integrated photonic circuits. These characteristics make them indispensable for next-generation optical communication networks, ultrafast signal processing, optical sensing, and emerging photonic computing technologies [8,9].

Modulational interaction refers to a fundamental nonlinear wave phenomenon in which a continuous or quasi-continuous wave propagating through a nonlinear dispersive medium loses its stability owing to the combined effects of dispersion and nonlinearity. Under appropriate physical conditions, small perturbations superimposed on the propagating wave undergo exponential growth rather than attenuation, causing the initially stable steady-state solution to become unstable. As a consequence, the continuous wave evolves into a temporally and spatially modulated structure, leading to efficient energy redistribution among different frequency components. This nonlinear process constitutes one of the most important mechanisms governing wave amplification, frequency conversion, and pattern formation in a broad range of physical systems, including optical fibers, semiconductor plasmas, fluid dynamics, Bose–Einstein condensates, and plasma media [10].

The instability associated with this process is commonly referred to as modulational instability (MI) or sideband instability. It represents a universal nonlinear phenomenon in which weak amplitude or phase perturbations imposed on a carrier wave are continuously amplified through nonlinear wave coupling. The nonlinear interaction transfers energy from the central carrier frequency to neighboring spectral components, resulting in the formation and growth of spectral sidebands. As the instability progresses, the continuous waveform gradually breaks into a sequence of localized pulses or wave packets, producing characteristic pulse-train structures. The simultaneous action of material nonlinearity and wave dispersion is therefore responsible for the onset and evolution of modulational instability, making it a key physical mechanism underlying nonlinear wave dynamics, pulse generation, optical amplification, and soliton formation in dispersive media [11–13].

The phenomenon of modulational instability was first identified and theoretically explained in the context of periodic surface gravity waves (Stokes waves) propagating on deep water. Benjamin and Feir demonstrated that a uniform train of water waves becomes unstable when subjected to small periodic perturbations, resulting in the exponential growth of sideband frequencies and the subsequent breakup of the original wave pattern [14]. This pioneering work established the theoretical foundation of modulational instability and led to the widespread recognition of the phenomenon as the Benjamin–Feir instability. Since its original discovery in hydrodynamics, the concept of modulational instability has evolved into a universal framework describing nonlinear wave propagation across numerous branches of physics, including nonlinear optics, plasma physics, condensed matter physics, semiconductor physics, and photonics, where it continues to play a central role in understanding wave amplification and nonlinear energy-transfer processes.

Modulational instability does not occur under all propagation conditions; rather, its onset is governed by a specific balance between the nonlinear response of the medium and its dispersive characteristics. The amplification of perturbations becomes possible only when the nonlinear phase shift induced by the optical field effectively compensates for the dispersive spreading of the wave. Among the various requirements for the occurrence of modulational instability, the presence of anomalous group-velocity dispersion (GVD) is one of the most significant. In the anomalous dispersion regime, optical pulses possessing shorter wavelengths (or higher frequencies) propagate with a higher group velocity than those with longer wavelengths. This unique dispersive behavior promotes phase matching between interacting waves and enables the exponential amplification of weak perturbations imposed on the carrier wave, thereby initiating the modulational instability process [15].

The above condition is generally considered in conjunction with a self-focusing Kerr-type optical nonlinearity, in which the refractive index of the medium exhibits a positive dependence on the intensity of the incident electromagnetic field. Under such circumstances, the optical field induces an intensity-dependent refractive index change, commonly referred to as the optical Kerr effect, resulting in nonlinear self-phase modulation and wave self-focusing. The combined action of anomalous dispersion and Kerr nonlinearity provides the necessary physical environment for the growth of modulation sidebands, thereby destabilizing the initially continuous optical wave and promoting the formation of localized pulse structures.

In its simplest theoretical description, modulational instability can be interpreted as the nonlinear interaction between a strong monochromatic carrier wave of angular frequency ω_0 and two weak perturbation sidebands located symmetrically at angular frequencies $\omega_0 + \omega$ and $\omega_0 - \omega$, where ω denotes the modulation frequency. The nonlinear coupling causes energy to be continuously transferred from the intense carrier wave to these sidebands, leading to their exponential amplification as the wave propagates through the nonlinear medium. This mechanism ultimately results in the breakup of the continuous-wave field into a train of temporally localized optical pulses.

From the viewpoint of nonlinear wave mixing, modulational instability represents a special case of the degenerate four-wave mixing (FWM) process. In this interaction, two photons (or quanta) associated with the pump wave at angular frequency $\boldsymbol{\omega}$ are annihilated simultaneously to generate a pair of new photons at the frequencies $\omega_0 + \omega$ and $\omega_0 - \omega$, while satisfying both energy conservation and phase-matching conditions. Consequently, modulational instability may be regarded as a parametric amplification process in which the nonlinear medium facilitates coherent energy exchange among the interacting waves. This interpretation establishes a direct connection between modulational instability and a broad class of nonlinear optical phenomena, including parametric amplification, frequency conversion, supercontinuum generation, optical soliton formation, and ultrafast pulse generation, thereby highlighting its fundamental importance in modern nonlinear optics and photonic technologies.

The selection of an appropriate nonlinear medium together with the operating optical frequency constitutes one of the most critical considerations in the design and optimization of high-performance optical modulators and other nonlinear photonic devices. The efficiency of modulational interactions, optical signal processing, and frequency conversion processes is strongly governed by the intrinsic nonlinear optical properties of the host material. Despite significant advances in photonic engineering, the realization of compact, efficient, and high-speed optical modulators continues to be largely constrained by material limitations. Consequently, the search for alternative nonlinear materials possessing superior optical characteristics and seamless compatibility with semiconductor fabrication technology has become an active area of research. This demand has been further intensified by the rapid advancement of planar photonic integrated circuits, high-capacity optical communication networks, and miniaturized optoelectronic systems, where device integration, scalability, and low-power operation are of paramount importance.

Among the various classes of nonlinear materials investigated to date, III–V compound semiconductors have emerged as one of the most promising candidates for advanced optoelectronic and photonic applications. These materials exhibit exceptional electronic and optical properties that make them highly suitable for the realization of efficient nonlinear optical devices and integrated photonic components [16–21].

Their technological importance arises from several unique physical characteristics, including:

1. High free-carrier concentration: The presence of a large density of free electrons or holes in doped III–V semiconductors significantly enhances carrier-induced nonlinear optical phenomena, thereby facilitating efficient nonlinear wave interactions.
2. Tailorable carrier dynamics: The carrier relaxation and recombination times can be precisely engineered through appropriate material design, heterostructure formation, and device architecture, allowing optimization of the nonlinear optical response for specific applications.
3. Strong resonant nonlinearities: These materials exhibit exceptionally large nonlinear optical susceptibilities in the vicinity of electronic band-gap transitions, resulting in efficient light–matter interaction and enhanced optical modulation.
4. Dual modulation mechanisms: Both the intensity-dependent refractive index and the optical absorption coefficient can be effectively manipulated, providing multiple mechanisms for controlling the propagation characteristics of electromagnetic waves.
5. Flexible device configurations: Optical devices based on III–V semiconductors can operate efficiently under normal-incidence illumination as well as within integrated waveguide geometries, offering considerable design flexibility for photonic circuits.
6. Excellent integrability: III–V semiconductor devices are readily compatible with existing optoelectronic platforms and can be monolithically integrated with lasers, photodetectors, optical amplifiers, modulators, and other functional photonic components, enabling the development of compact and multifunctional integrated photonic systems.

It is well established that both resonant and non-resonant nonlinear optical effects exhibited by III–V semiconductor materials—including InSb, GaAs, GaSb, InAs, and related compounds—have been extensively exploited for the development of a wide variety of optoelectronic devices, such as optical switches, modulators, detectors, infrared sensors, frequency converters, and high-speed communication components [22]. Furthermore, continuous improvements in device performance have been achieved through deliberate engineering of material properties using techniques such as controlled doping, compositional tuning, heterostructure design, quantum confinement, micro- and nanostructuring, and band-gap engineering [22]. These approaches provide effective means for tailoring the electronic structure, carrier transport properties, and nonlinear optical response of semiconductor materials to satisfy the requirements of modern photonic technologies.

An additional advantage of III–V semiconductors lies in the fact that their nonlinear optical characteristics can be actively controlled by externally applied electric and magnetic fields [23–25]. Such external perturbations modify the carrier dynamics, plasma oscillations, cyclotron motion, and dielectric response of the semiconductor, thereby providing an efficient mechanism for tuning its optical susceptibility and nonlinear interaction strength. The magnetic-field dependence of the nonlinear response is particularly important in magnetoactive semiconductors, where the electron cyclotron frequency can be adjusted over a wide range, allowing dynamic control over wave propagation and nonlinear coupling processes. Similarly, externally applied electric fields influence carrier transport and polarization effects, offering additional flexibility in the modulation of optical properties.

The field-dependent tunability of nonlinear optical parameters offers a valuable platform for investigating the fundamental mechanisms governing modulational interactions, wave amplification, and nonlinear energy-transfer processes in semiconductor plasmas. By appropriately selecting the material composition, doping concentration, and external biasing conditions, it becomes possible to optimize the efficiency of modulational instability and related nonlinear optical phenomena. In view of the outstanding optical, electronic, and magneto-optical characteristics of III–V semiconductor crystals, comprehensive theoretical and experimental investigations of modulational interactions in these materials are of considerable scientific and technological importance. Such studies become even more significant in media that support multiple optical excitation modes, where the coexistence of plasmons, phonons, and electromagnetic waves leads to complex energy exchange mechanisms accompanied by controllable gain or loss. A deeper understanding of these nonlinear interactions is expected to contribute substantially to the development of next-generation optoelectronic devices, tunable photonic systems, high-speed optical communication technologies, and semiconductor-based nonlinear integrated circuits.

III–V compound semiconductors possess a rich spectrum of elementary excitations that play a decisive role in determining their optical, electrical, and transport characteristics. Owing to their unique electronic band structure and strong carrier–lattice interactions, these materials support several collective excitation modes, including plasmons (PLs), acoustical phonons (APs), optical phonons (OPs), excitons, magnons, and other quasiparticles. The coexistence of these excitation modes gives rise to a variety of plasmon–phonon coupling mechanisms, which constitute one of the most important many-body interaction phenomena in semiconductor physics. Such interactions significantly influence carrier dynamics, dielectric response, optical absorption, electrical conductivity, energy relaxation, and nonlinear optical behavior, thereby playing a crucial role in the operation and optimization of semiconductor-based optoelectronic devices.

Among the different plasmon–phonon interactions, the coupling between plasmons and acoustical phonons has attracted considerable scientific interest because of its profound influence on nonlinear wave propagation and carrier transport. In non-centrosymmetric III–V semiconductor crystals, the absence of inversion symmetry enables the occurrence of the piezoelectric effect, whereby a mechanically induced strain generates an internal electric polarization and the corresponding electric field. Consequently, propagating acoustical phonons produce time-varying piezoelectric fields that interact strongly with the free carriers present in the semiconductor. Depending on the crystallographic orientation and propagation direction, acoustical phonons may become piezoelectrically active, resulting in efficient coupling between lattice vibrations and the electron plasma.

The piezoelectric interaction between electrons and acoustical phonons has a pronounced impact on both the optical and electronic properties of semiconductor materials. This interaction modifies the carrier mobility, scattering rate, dielectric response, and nonlinear optical susceptibility, thereby influencing the propagation characteristics of electromagnetic waves within the crystal. As a result, piezoelectric electron–phonon coupling provides an effective mechanism for controlling nonlinear optical processes and enhancing the performance of various photonic and optoelectronic devices. In particular, the interaction facilitates efficient modulation of optical signals by enabling dynamic control over the refractive index and the optical phase of propagating electromagnetic waves.

One of the most significant consequences of this interaction is the acousto-optic effect, in which incident optical radiation is scattered by acoustical phonons or low-frequency electromagnetic waves. The periodic strain associated with the acoustic wave creates a moving refractive-index grating inside the semiconductor, causing diffraction of the incident light. This phenomenon offers a highly efficient means of manipulating the frequency, intensity, phase, polarization, and propagation direction of an optical beam through external acoustic excitation [26]. Because the acoustic wave can be precisely controlled, acousto-optic modulation provides excellent tunability, high modulation speed, and reliable operation, making it an indispensable technique in modern photonics.

The ability to dynamically modulate optical radiation using acoustical phonons has led to numerous technological applications in optical communication, information transmission, signal processing, laser beam steering, optical imaging, spectrum analysis, and display technologies [27]. Furthermore, nonlinear plasmon–acoustic phonon interactions constitute the physical basis for the operation of several important semiconductor devices, including acousto-optic modulators, optical deflectors, tunable frequency shifters, optical filters, delay lines, and low-noise microwave and optical amplifiers [28,29]. In these devices, an incident pump beam interacts coherently with an acoustical phonon or a low-frequency electromagnetic wave, enabling efficient energy transfer and controlled modulation of the optical signal through nonlinear wave coupling.

The strong coupling between plasmons and piezoelectrically active acoustical phonons therefore provides an effective platform for investigating nonlinear wave interactions, modulational instability, and optical amplification in III–V semiconductor materials. A detailed understanding of these coupled excitation processes is essential for the development of next-generation semiconductor photonic devices, particularly those requiring high-speed modulation, low-power operation, tunable optical functionality, and seamless integration with advanced optoelectronic circuits.

Over the past several decades, modulational amplification of optically excited coherent collective excitations in III–V semiconductor materials has attracted considerable attention owing to its fundamental significance in nonlinear optics and its potential applications in advanced optoelectronic technologies. Extensive theoretical and experimental investigations have demonstrated that nonlinear interactions among electromagnetic waves, free-carrier plasma oscillations, and lattice vibrations can give rise to efficient wave amplification, parametric energy transfer, and modulation instability in semiconductor media. Consequently, numerous research groups have explored the underlying physical mechanisms governing these nonlinear processes, thereby establishing III–V semiconductors as promising platforms for the realization of high-speed photonic and optoelectronic devices [30–32].

Among the important contributions in this field, Neogi [33] presented a comprehensive theoretical investigation of acousto-optic modulation of an intense electromagnetic wave propagating through a strain-dependent semiconductor crystal containing excess free charge carriers. In this study, the influence of carrier diffusion was explicitly incorporated into the theoretical model, enabling a more realistic description of the interaction between the optical pump wave and acoustically induced strain fields. The analysis demonstrated that carrier diffusion plays a significant role in determining the efficiency of acousto-optic modulation by modifying the carrier-density distribution, dielectric response, and nonlinear coupling between the interacting optical and acoustic waves.

Subsequently, Ghosh and Rishi [34,35] carried out detailed investigations of acousto-optic modulation and demodulation in magnetoactive diffusive semiconductor plasmas. Their studies focused on the combined influence of externally applied magnetic fields and carrier diffusion on the nonlinear interaction between electromagnetic radiation and acoustical waves. The presence of the magnetic field was shown to alter the carrier dynamics through cyclotron motion, thereby modifying the nonlinear susceptibility and the efficiency of wave coupling. Their results revealed that magnetic-field-induced anisotropy provides an effective means of controlling modulation characteristics and enhancing the performance of semiconductor-based acousto-optic devices.

More recently, Nimje et al. [36] investigated the influence of carrier heating on both the steady-state and transient amplification characteristics of modulated waves generated in magnetoactive semiconductor plasmas. Their theoretical analysis considered the increase in carrier temperature resulting from the absorption of high-intensity electromagnetic radiation and examined its effect on nonlinear plasma dynamics. The study demonstrated that carrier heating significantly modifies important plasma parameters, including carrier mobility, collision frequency, dielectric response, and nonlinear optical susceptibility, which in turn influence the gain characteristics, temporal evolution, and amplification efficiency of the modulated optical wave. These findings highlighted the importance of incorporating thermal effects into theoretical models for accurately predicting the nonlinear optical behavior of semiconductor plasmas under high-power optical excitation.

Collectively, these investigations have substantially advanced the understanding of nonlinear wave interactions, acousto-optic modulation, and modulational amplification in III–V semiconductor materials. Nevertheless, the complex interplay among plasmons, acoustical phonons, carrier diffusion, magnetic-field effects, and nonlinear optical susceptibilities continues to present important scientific challenges. A comprehensive theoretical framework that simultaneously accounts for these coupled phenomena remains essential for optimizing modulational amplification processes and for the design of next-generation semiconductor-based photonic, optoelectronic, and nonlinear integrated devices.

A comprehensive review of the existing literature indicates that theoretical and experimental studies on the modulational amplification of optically excited coherent collective modes in III–V semiconductor crystals, including InSb, GaAs, GaSb, InAs, and related compounds, remain relatively limited despite the considerable technological importance of these materials. Although substantial progress has been made in understanding nonlinear optical phenomena in semiconductor media, detailed investigations concerning the nonlinear coupling between collective plasma oscillations and lattice vibrations, particularly under the influence of externally applied magnetic fields, are still scarce. This gap in the literature underscores the necessity for developing comprehensive theoretical models capable of elucidating the fundamental mechanisms governing nonlinear wave interactions and modulational amplification in magnetoactive semiconductor plasmas.

Most III–V semiconductor crystals belong to the class of weakly piezoelectric, non-centrosymmetric materials, in which the piezoelectric interaction between free carriers and lattice vibrations plays a dominant role in determining their nonlinear optical response. Owing to their relatively weak piezoelectric constants, the interaction of an incident pump electromagnetic wave with low-frequency acoustical phonons (APs) constitutes the predominant scattering mechanism responsible for nonlinear energy exchange within these materials [37]. The strain-induced electric field associated with piezoelectrically active acoustical phonons provides an efficient coupling channel between electromagnetic waves and free-carrier plasma oscillations, thereby significantly influencing the optical and transport properties of the semiconductor. Consequently, a rigorous theoretical understanding of plasmon–acoustical phonon (PL–AP) interactions is essential for optimizing the performance of semiconductor-based optical modulators and other nonlinear photonic devices.

Motivated by the scientific and technological significance of these interactions, the present work develops a comprehensive analytical framework to investigate modulational amplification arising from plasmon–acoustical phonon coupling in weakly piezoelectric, magnetoactive, heavily doped III–V semiconductor crystals. The theoretical analysis is formulated using the hydrodynamic model of a one-component semiconductor plasma, which provides a self-consistent description of the collective dynamics of free electrons interacting with electromagnetic and acoustic fields. The nonlinear coupling between the interacting waves is analyzed within the framework of the coupled-mode theory, enabling the derivation of analytical expressions describing the growth of modulation sidebands and the corresponding amplification characteristics.

In the proposed model, the origin of optical nonlinearity is attributed to the piezoelectric interaction between electron plasma waves (plasmons) and low-frequency acoustical phonons generated through lattice vibrations. The propagating acoustic wave induces a periodic strain field that generates an internal piezoelectric electric field, which interacts with the free-electron plasma and modulates the dielectric properties of the semiconductor. Under the influence of an externally applied magnetic field, this acoustic field becomes strongly coupled with the propagating optical modulation wave. When the interacting waves satisfy the appropriate phase-matching conditions, coherent energy transfer occurs from the pump wave to the modulation sidebands, resulting in the amplification of the modulated optical signal through a parametric nonlinear interaction.

The modulational amplification process is quantitatively characterized by the effective third-order optical susceptibility, $\chi^{(3)}$, of the semiconductor medium. This nonlinear susceptibility represents the strength of the coupling between the optical field, the free-electron plasma, and the acoustical phonons, thereby determining the efficiency of nonlinear wave mixing and modulation amplification. The magnitude of $\chi^{(3)}$ is strongly influenced by material parameters such as carrier concentration, piezoelectric coupling strength, plasma frequency, collision frequency, and externally applied magnetic field, making it an important parameter for tailoring the nonlinear optical response of semiconductor devices.

It is important to distinguish the nonlinear mechanism considered in the present investigation from the conventional electro-optic Kerr effect. In the classical Kerr effect, the nonlinear response originates primarily from the interaction of the optical field with bound electrons, resulting in an intensity-dependent polarization that is inherently local in nature. The induced nonlinear polarization at any point within the material depends only on the optical field existing at that particular location. In contrast, the nonlinear interaction investigated in the present work corresponds to the piezoelectric Kerr effect, where the nonlinearity arises predominantly from the collective motion of free charge carriers coupled with acoustically induced piezoelectric fields. Because the carrier transport involves finite spatial motion and collective plasma dynamics, the resulting nonlinear polarization exhibits a nonlocal character, extending over distances comparable to the carrier transport length. This nonlocality introduces additional degrees of freedom into the modulational interaction and significantly alters the amplification characteristics compared with conventional Kerr-type nonlinear optical processes.

Therefore, the present theoretical investigation aims to provide a deeper understanding of the nonlinear dynamics associated with plasmon–acoustical phonon interactions in weakly piezoelectric magnetoactive III–V semiconductors. The results are expected to contribute to the optimization of modulational amplification processes and to facilitate the development of highly efficient semiconductor-based optical modulators, nonlinear photonic components, tunable microwave–optical devices, and next-generation integrated optoelectronic systems.

To validate the proposed theoretical formulation and evaluate its practical applicability, an extensive numerical investigation has been carried out using a set of material parameters corresponding to the weakly piezoelectric III–V semiconductor crystal n-type indium antimonide (n-InSb). The numerical simulations are performed under realistic experimental conditions by considering the semiconductor crystal to be irradiated with a frequency-doubled, pulsed 10.6 μm CO_2 laser, which serves as the optical pump source for exciting the nonlinear plasmon–acoustical phonon interaction. The selected material system is particularly suitable for the present study because of its high electron mobility, low effective electron mass, strong magneto-optical response, and pronounced free-carrier nonlinearities, all of which make n-InSb an excellent candidate for investigating modulational amplification in magnetoactive semiconductor plasmas.

The numerical analysis has been undertaken to examine the influence of various physical parameters on the onset and evolution of the modulational amplification process and to verify the predictions of the analytical model. In particular, the dependence of the threshold pump-field amplitude, modulation growth rate, and nonlinear amplification characteristics on key material and external control parameters has been systematically evaluated. The obtained results not only demonstrate the consistency and validity of the developed theoretical framework but also provide valuable physical insight into the mechanisms governing plasmon–acoustical phonon coupling in weakly piezoelectric III–V semiconductor crystals. Furthermore, the numerical findings establish the suitability of n-InSb as a promising nonlinear medium for the realization of efficient magnetically tunable optical modulators and other advanced semiconductor-based photonic and optoelectronic devices.

2. Theoretical formulations

For the present theoretical investigation, the classical hydrodynamic model of a homogeneous n-type semiconductor plasma operating at 77 K (liquid nitrogen temperature) has been employed to describe the nonlinear interaction between electromagnetic waves and collective plasma oscillations. The semiconductor is assumed to contain electrons as the majority charge carriers and is subjected simultaneously to an intense optical pump field and a uniform externally applied magnetostatic field while remaining under thermal equilibrium conditions. Within this framework, the hydrodynamic description provides an effective means of analyzing the modulational interaction and the resulting amplification of modulation sidebands in a weakly piezoelectric, magnetoactive, heavily doped III–V semiconductor crystal. The nonlinear optical response responsible for this amplification is characterized by the effective third-order optical susceptibility, $\chi^{(3)}$, which originates from the piezoelectric coupling between electron plasma oscillations and low-frequency acoustical phonons.

The hydrodynamic approach is particularly appropriate for the present analysis because it provides a physically transparent and mathematically tractable description of collective carrier dynamics without sacrificing the essential physics governing the nonlinear interaction. Instead of treating the motion of individual electrons through a microscopic kinetic formalism, the electron ensemble is modeled as a compressible charged fluid whose behavior is described by a limited set of macroscopic variables, including the average carrier concentration, drift velocity, momentum, pressure, and current density. This continuum formulation considerably simplifies the analytical treatment of coupled plasma–phonon interactions while accurately capturing the collective response of the semiconductor plasma to external electromagnetic and magnetic fields.

The validity of the hydrodynamic approximation is restricted to the long-wavelength regime, where the characteristic wavelength of the interacting electromagnetic disturbance is significantly larger than the mean free path of the charge carriers. Mathematically, this requirement may be expressed as:

$$kl \ll 1,$$

where (k) denotes the wave number of the propagating disturbance and (l) represents the mean free path of the electrons. Under this condition, carrier transport is governed predominantly by collective fluid-like behavior rather than individual particle kinetics, thereby justifying the application of the hydrodynamic model to the present problem. Consequently, the derived analytical expressions remain valid only within this long-wavelength approximation, where nonlocal kinetic effects can be neglected without introducing significant inaccuracies.

When a semiconductor crystal is subjected to an external magnetostatic field, its electromechanical and optical properties become anisotropic because the magnetic field modifies the motion of free carriers through the Lorentz force. As a result, the piezoelectric coupling coefficients and the associated dielectric susceptibility can no longer be treated as isotropic quantities. In general, the piezoelectric tensor acquires non-zero off-diagonal components, reflecting the anisotropic coupling between mechanical deformation, electric polarization, and carrier motion. This tensorial behavior plays an important role in determining the propagation characteristics of coupled electromagnetic and acoustic waves in magnetoactive semiconductor media.

To facilitate the analytical development while retaining the essential physical features of the interaction, the present study considers the generation and propagation of a longitudinal acoustical wave in a cubic III–V semiconductor crystal possessing $\bar{4}3m$ crystallographic symmetry. Owing to the high symmetry of this crystal class and the chosen propagation geometry, the piezoelectric tensor simplifies considerably and can effectively be represented by a single independent piezoelectric coefficient [38,39]. This simplification substantially reduces the mathematical complexity of the coupled-wave analysis while preserving the dominant piezoelectric interaction responsible for the nonlinear coupling between plasmons, acoustical phonons, and the optical pump field. Consequently, the adopted theoretical model provides an efficient and rigorous framework for investigating modulational amplification in weakly piezoelectric magnetoactive III–V semiconductor crystals under experimentally relevant conditions.

2.1 *Effective third-order optical susceptibility*

For the theoretical formulation of the proposed model, we consider the propagation of a spatially uniform, linearly polarized electromagnetic pump wave $\vec{E}_0 = \hat{x}E_0 \exp(-i\omega_0 t)$ through a weakly piezoelectric, heavily doped III–V semiconductor crystal subjected to an externally applied static magnetic field $\vec{B}_0 = \hat{z}B_0$. The pump electromagnetic wave is assumed to propagate along the x-axis, while the uniform magnetostatic field is applied along the z-axis, thereby establishing a mutually orthogonal configuration between the propagation direction of the optical field and the external magnetic field. Such a transverse-field geometry is frequently employed in magneto-optical studies because it provides efficient coupling between the electromagnetic wave, the free-electron plasma, and the acoustically induced piezoelectric field, thereby facilitating nonlinear wave interactions.

The optical pump wave is assumed to possess a spatially uniform ($|\vec{k}_0| \approx 0$) electric-field amplitude, implying that its variation over the characteristic interaction region is negligible. This approximation is justified within the framework of the electric dipole approximation, where the wavelengths associated with the optically excited collective modes—such as electron plasma waves and low-frequency acoustical phonons—are considerably smaller than the characteristic spatial scale over which the pump-field amplitude changes [40]. Consequently, the pump field may be regarded as locally homogeneous throughout the interaction volume.

Mathematically, this assumption corresponds to the condition that the spatial gradient of the pump field is negligibly small, allowing the spatial derivative of the pump amplitude to be ignored in the governing equations. The electromagnetic pump therefore acts as a uniform driving field that supplies energy to the nonlinear interaction without introducing additional spatial modulation. This approximation significantly simplifies the analytical treatment of the coupled-wave equations while preserving the essential physical mechanisms responsible for modulational amplification. Furthermore, it enables the nonlinear coupling between the optical pump, the electron plasma oscillations, and the piezoelectrically active acoustical phonons to be investigated without the complications arising from pump-beam diffraction or spatial inhomogeneity.

The adopted propagation geometry and the assumption of a spatially uniform pump field therefore provide a convenient and physically realistic framework for analyzing the nonlinear plasmon–acoustical phonon interaction in magnetoactive III–V semiconductor crystals. Under these conditions, the influence of the externally applied magnetic field on the carrier dynamics and the resulting third-order nonlinear optical response can be investigated systematically, leading to a clearer understanding of the modulational amplification process and its dependence on the material and external control parameters.

The underlying physical mechanism responsible for the proposed modulational amplification process may be understood in terms of the nonlinear coupling between the optical pump field, electron plasma oscillations (plasmons), and acoustical phonons within the weakly piezoelectric semiconductor medium. When an intense electromagnetic pump wave propagates through the magnetoactive III–V semiconductor crystal, it exerts a nonlinear force on the free-electron plasma, resulting in a periodic redistribution of the carrier density. Through the piezoelectric effect, this carrier-density modulation generates an oscillatory strain field that excites low-frequency acoustical phonons. Consequently, a dynamic coupling is established between the collective oscillations of the free carriers, represented by the electron-plasma wave (plasmon), and the lattice vibrations associated with the acoustical phonons.

The interaction between these two collective excitation modes gives rise to a space-charge field produced by the periodic separation and redistribution of free charge carriers. This internally generated electric field enhances the coupling between the electromagnetic wave and the acoustic disturbance, thereby driving the acoustical wave at the modulation frequency. As the interaction evolves, coherent energy is transferred from the intense optical pump to the coupled plasmon–acoustical phonon system. Under appropriate phase-matching conditions, this energy transfer becomes highly efficient, resulting in the exponential growth of the modulation sidebands and the amplification of the acoustic wave.

The nonlinear coupling is further strengthened by the presence of the externally applied magnetic field, which modifies the motion of free carriers through the Lorentz force and alters the dielectric response of the semiconductor plasma. The combined influence of the magnetic field, piezoelectric interaction, and carrier dynamics significantly enhances the effective third-order nonlinear optical susceptibility of the medium, thereby increasing the efficiency of the modulational interaction.

As a consequence, both the optical modulation wave and the acoustical wave experience substantial amplification through a nonlinear optical pumping mechanism. In this process, the intense pump beam serves as the primary energy source, while the plasmon–acoustical phonon interaction provides the coupling channel that enables efficient energy transfer to the modulation sidebands. The resulting nonlinear amplification process forms the physical basis for the operation of high-performance acousto-optic and magneto-optic devices and demonstrates the potential of weakly piezoelectric III–V semiconductor crystals as promising media for advanced optical modulation, signal amplification, and integrated photonic applications.

Accordingly, the fundamental equations governing the present modulational interaction are given as follows:

$$\frac{\partial \vec{v}_0}{\partial t} + \nu \vec{v}_0 = -\frac{e}{m}[\vec{E}_0 + (\vec{v}_0 \times \vec{B}_0)] = -\frac{e}{m}\vec{E}_e \tag{1}$$

$$\frac{\partial \vec{v}_1}{\partial t} + \nu \vec{v}_1 + \left(\vec{v}_0 . \frac{\partial}{\partial x}\right)\vec{v}_1 = -\frac{e}{m}[\vec{E}_1 + (\vec{v}_1 \times \vec{B}_0)] \tag{2}$$

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} + n_1 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial n_1}{\partial x} = 0 \tag{3}$$

Equations (1) and (2) represent the momentum balance equations corresponding to the pump wave and the generated (product) waves, respectively. In these equations, $\vec{v}_0$ and $\vec{v}_1$ denote the oscillatory electron fluid velocities associated with the pump electromagnetic field and the product-wave field, respectively. The parameter ν represents the phenomenological electron–electron momentum relaxation (collision) frequency, while *m** and -*e* denote the effective mass and charge of the electron, respectively.

In Eq. (1), $\vec{E}_e$ corresponds to the effective electric field experienced by the charge carriers, which includes the contribution of the Lorentz force ($\vec{v}_0 \times \vec{B}_0$) arising from the externally applied static magnetic field. In contrast, for Eq. (2), the Lorentz force associated with the perturbed magnetic field ($\vec{v}_0 \times \vec{B}_1$) has been neglected by assuming that the propagating acoustical phonon generates only a longitudinal electric field, thereby making the magnetic perturbation insignificant.

It is important to note that the optical nonlinearity governing the carrier dynamics in the semiconductor plasma originates primarily from the convective acceleration term $(\vec{v}.\nabla)\vec{v}$ and the Lorentz force $-e(\vec{v}_{0,1} \times \vec{B}_0)$, both of which depend on the total optical intensity of the incident electromagnetic field. These nonlinear terms are responsible for the coupling between the interacting waves and consequently lead to the modulational amplification process.

Equation (3) represents the electron continuity equation, which expresses the conservation of charge within the semiconductor plasma. Here, n_0 and n_1 denote the equilibrium and perturbed electron number densities, respectively.

According to the multimode theory of modulational interactions, the intense optical pump wave excites an acoustical phonon (AP) mode at the acoustic frequency (ω_{ap}) through lattice vibrations within the weakly piezoelectric semiconductor crystal. These lattice vibrations generate a periodic strain field, which, via the piezoelectric effect, induces a corresponding perturbation in the free-electron density. The resulting electron-density modulation interacts nonlinearly with the incident pump wave, thereby establishing a coherent coupling between the optical, plasma, and acoustic modes. Under suitable phase-matching conditions, this nonlinear coupling drives the acoustical wave at the modulation frequency and facilitates the transfer of energy from the pump wave to the generated sidebands, leading to modulational amplification.

The propagation of the acoustical wave in the weakly piezoelectric III–V semiconductor crystal is governed by the following equation of motion:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\Gamma_{ap}\frac{\partial u(x,t)}{\partial t} + \frac{\beta}{\rho}\frac{\partial E_s}{\partial x} = \frac{C}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{4}$$

where $u(x,t) = u\exp[i(k_{ap}x - \omega_{ap}t)]$ denotes the displacement of lattice points from their mean position under the influence of interfering electromagnetic fields. ρ, β, Γ_{ap} and C represent the mass density of the crystal, piezoelectric constant, phenomenological damping parameter of AP mode and crystal elastic constant, respectively. It is assumed that the acoustic wave frequency is much lower than the pump wave frequency (i.e. $\omega_{ap} \ll \omega_0$).

The redistribution and migration of free charge carriers under the influence of the optical and acoustic fields result in the accumulation of positive and negative charges in different regions of the semiconductor crystal. This charge separation gives rise to an internal space-charge electric field, E_1, which plays a crucial role in mediating the nonlinear coupling between the electron plasma and the acoustical phonons. The generated space-charge field enhances the interaction among the coupled waves and contributes significantly to the modulational amplification process.

The expression for the space-charge electric field can be obtained by simultaneously solving the continuity equation [Eq. (3)] together with Poisson's equation [Eq. (5)], yielding

$$\frac{\partial E_1}{\partial x} + \frac{\beta}{\varepsilon}\frac{\partial^2 u}{\partial x^2} = -\frac{n_1 e}{\varepsilon}. \tag{5}$$

Here, $\varepsilon(=\varepsilon_0\varepsilon_1)$ denotes the scalar dielectric constant of the semiconductor medium, while ε_1 represents its static dielectric constant. The space-charge field developed within the crystal is therefore governed by the dielectric response of the medium and the redistribution of free charge carriers.

For efficient nonlinear coupling and energy transfer among the interacting waves, the proposed modulational process must satisfy the phase-matching (or momentum and energy conservation) conditions, which ensure coherent amplification of the modulation sidebands. These conditions are expressed as follows:

$\hbar\vec{k}_{\pm} = \hbar\vec{k}_0 \pm \hbar\vec{k}_{ap}$ (momentum conservation relation), and

$\hbar\omega_{\pm} = \hbar\omega_0 \pm \hbar\omega_{ap}$ (energy conservation relation).

For the case of spatially uniform laser irradiation, the pump field is assumed to remain constant throughout the interaction region, i.e., $|\vec{k}_{\pm}| \approx |\vec{k}_0 \pm \vec{k}_{ap}| \approx |\vec{k}_1| = k$.

The carrier-density perturbation generated by the intense optical pump is coupled to the acoustical phonon (AP) mode and oscillates at the corresponding acoustic frequency (ω_{ap}). As a result of this periodic density modulation, the pump wave undergoes nonlinear modulation, giving rise to scattered waves at the upper and lower sideband frequencies, $\omega_{+}(=\omega_0 + p\omega_{ap})$, where p is an integer denoting the order of the sideband.

In the present analysis, only the first-order resonant sidebands (p = 1) are considered. Furthermore, by assuming a sufficiently long interaction length (equivalently, an infinitely long crystal), the higher-order sidebands (p > 1) become off-resonant and their contributions to the nonlinear interaction are negligibly small. Consequently, the modulational process is governed primarily by the coupling between the incident pump wave at frequency ω_0 and the first-order scattered waves at frequencies $\omega_0 \pm \omega_{ap}$, which interact efficiently with the acoustical phonon mode under the prescribed phase-matching conditions.

The governing equation for the carrier-density fluctuations associated with the coupled electron–plasma wave (plasmon) in a weakly piezoelectric, magnetoactive, n-type doped semiconductor is derived under the collision-dominated regime ($\nu \gg kv_0, \omega$). The derivation is based on the simultaneous use of Eqs. (1)–(5) together with the linearized perturbation approach, wherein only first-order perturbation terms are retained while higher-order nonlinear contributions are neglected. This approximation provides a self-consistent description of the collective plasma response to the applied optical and acoustic fields and forms the basis for analyzing the modulational amplification process. Accordingly, the resulting equation governing the carrier-density perturbation is expressed as follows:

$$\frac{\partial^2 n_1(\omega_{\pm},k_{\pm})}{\partial t^2} + \nu\frac{\partial n_1(\omega_{\pm},k_{\pm})}{\partial t} + \bar{\omega}_p^2 n_1(\omega_{\pm},k_{\pm}) + \frac{(e/m)\beta n_0}{\varepsilon}\frac{\partial^2 u}{\partial x^2} = -\bar{E}\frac{\partial n_1(\omega_{\pm},k_{\pm})}{\partial x}, \tag{6}$$

where $\bar{E} = -\left(\frac{e}{m}E_0 + \omega_0 v_{0y}\right)$,

$\bar{\omega}_p^2 = \frac{\nu^2\omega_p^2}{(\nu^2 + \omega_c^2)}$, in which

$\omega_c = \frac{eB_0}{m}$ (electron-cyclotron frequency), and

$\omega_p = \left(\frac{n_0 e^2}{m\varepsilon}\right)^{1/2}$ (electron-plasma frequency).

In deriving Eq. (6), the Doppler shift has been neglected under the assumption $\omega_0 \gg \nu > kv_0$. Furthermore, both the pump electric-field component along the (y)-direction and the pump magnetic field have been ignored by assuming that their effects on the nonlinear interaction are insignificant compared with the dominant longitudinal field components.

The carrier-density perturbations generated through the nonlinear interaction oscillate at the frequencies of the forced waves, namely the upper and lower sidebands. Accordingly, the density fluctuations corresponding to these sideband frequencies can be expressed as:

$$n_1(\omega_+, k_+) = \frac{i(e/m)\beta^2 k^3 n_0 (X_+)^{-1} E_{ap}}{\varepsilon\rho\Omega_{ad}^2}, \tag{7a}$$

and

$$n_1(\omega_-, k_-) = \frac{i(e/m)\beta^2 k^3 n_0 (X_-)^{-1} E_{ap}}{\varepsilon\rho\Omega_{ad}^2}. \tag{7b}$$

Here, $\Omega_{ad}^2 = \omega_{ap}^2 - kv_{ap}^2 + 2i\Gamma_{ap}\omega_{ap}$ represents the AP dispersion.

$X_+ = \bar{\omega}_p^2 - \omega_+^2 - i\nu\omega_+ + ik\bar{E}$, and

$X_- = \bar{\omega}_p^2 - \omega_-^2 - i\nu\omega_- + ik\bar{E}$, in which

$\omega_+ = \omega_{ap} + \omega_0$ and $\omega_- = \omega_{ap} - \omega_0$ represent the frequencies corresponding to upper and lower side bands, respectively. Here, $v_{ap}[= (C/\rho)^{1/2}]$ denotes the velocity of the acoustical phonon (AP) mode in the semiconductor medium. In deriving Eqs. (7a) and (7b), it has further been assumed that the electron-density perturbations $n_1(\omega_\pm, k_\pm)$ associated with the upper and lower sideband frequencies vary harmonically as $\exp[i(\omega_\pm t - k_\pm x)]$.

Equation (7) clearly indicates that the sideband waves are nonlinearly coupled to the acoustical phonon mode through the induced carrier-density perturbations in the presence of a strong optical pump field. It is also evident from these expressions that the amplitudes of the density perturbations, $n_1(\omega_\pm, k_\pm)$, are directly influenced by the pump-field amplitude (via $\bar{E}$).

The carrier-density fluctuations generated at the sideband frequencies modify the dielectric response of the semiconductor and consequently influence the propagation characteristics of the scattered electromagnetic waves. These effects can be investigated by employing the general electromagnetic wave equation.

In the present analysis of modulational interaction in a magnetoactive doped semiconductor crystal, the contribution of the transition dipole moment has been neglected in order to isolate and examine the role of the nonlinear current density in the generation of the medium's nonlinear polarization. Under this assumption, the induced polarization arises predominantly from the collective motion of free charge carriers rather than from bound-electron transitions.

Accordingly, the nonlinear current densities associated with the upper and lower sideband waves are obtained as follows:

$$J(\omega_+) = -en_1(\omega_+)v_0 \tag{8a}$$

and

$$J(\omega_-) = -en_1(\omega_-)v_0^*. \tag{8b}$$

Henceforth, the nonlinear polarization induced at the modulated (sideband) frequencies, $P(\omega_\pm)$, is obtained by considering it as the time integral of the corresponding nonlinear current density, $J(\omega_\pm)$. Using this relationship, the induced nonlinear polarization can be expressed as follows:

$$P(\omega_\pm) = \int J(\omega_\pm)dt. \tag{9}$$

The effective nonlinear polarization associated with the modulated wave is obtained by introducing the effective third-order nonlinear optical susceptibility, which relates the induced nonlinear polarization to the interacting electromagnetic fields. Accordingly, the effective nonlinear polarization may be defined as follows:

$$P_e = P(\omega_+) + P(\omega_-). \tag{10}$$

For efficient amplification of the modulated waves, it is necessary that the upper and lower sidebands contribute equally to the nonlinear interaction. Under this condition, the energy exchanged between the pump wave and the sidebands is transferred coherently to the acoustical phonon (AP) mode through the piezoelectric coupling mechanism. Consequently, the acoustic wave undergoes amplification, leading to enhanced modulational growth and efficient nonlinear energy transfer within the semiconductor medium.

By combining Eqs. (7)–(10) and accounting for the contributions from both the upper and lower sidebands, the total effective nonlinear polarization of the modulated wave can be obtained. This effective polarization incorporates the combined influence of the carrier-density perturbations and the nonlinear current densities generated during the modulational interaction. Accordingly, the resulting expression for the total effective polarization is given by

$$P_e = \frac{-i(e/m)\beta^2 k^3 v_{ap}\omega_p^2\omega_0 (X_1)|E_0||E_{ap}|}{\rho\Omega_{ad}^2(\omega_0^2 - \omega_c^2)}, \tag{11}$$

where

$$X_1 = \frac{1}{\omega_+}(\Omega_+^2 - i\nu\omega_+ + ik\bar{E})^{-1} - \frac{1}{\omega_-}(\Omega_-^2 - i\nu\omega_- + ik\bar{E})^{-1},$$

in which

$\Omega_+^2 = \bar{\omega}_p^2 - \omega_+^2$, and $\Omega_-^2 = \bar{\omega}_p^2 - \omega_-^2$.

Equation (11) demonstrates that the effective third-order nonlinear susceptibility, P_e, provides the coupling between the carrier-density perturbations oscillating at the sideband frequencies $\omega_\pm$ and the acoustical phonon (AP) mode at the frequency ω_{ap}. This nonlinear coupling is responsible for the transfer of energy from the optical pump to the modulated waves, thereby governing the modulational amplification process.

Further algebraic simplification of Eq. (11) yields the following expression for the third-order nonlinear polarization, $P_e^{(3)}$:

$$P_e^{(3)} = \frac{2(e/m)^2\beta^2 k^4 v_{ap}\omega_p^2(\Omega^2-\nu^2)(X_2)^{-1}|E_0|^2|E_{ap}|}{\rho\Omega_{ad}^2(\omega_0^2-\omega_c^2)^2}, \quad (12)$$

where $X_2 = \left(\Omega^2+\nu^2-\frac{k^2\bar{E}^2}{\omega_0^2}\right)^2 + \frac{4k^2\Omega^2\bar{E}^2}{\omega_0^2}$,

in which $\Omega^2 = \omega_0^2 - \bar{\omega}_p^2$.

The transverse components of the oscillatory electron fluid velocity, $\vec{v}_0$, in the presence of the optical pump field $\vec{E}_0$ and the externally applied static magnetic field $\vec{B}_0$, can be obtained by solving the electron momentum equation [Eq. (1)]. These velocity components describe the transverse motion of the free carriers under the combined influence of the electric and magnetic fields and play an important role in determining the nonlinear current density responsible for the modulational interaction. Accordingly, the transverse components of the electron fluid velocity are given by

$$v_{0x} = \frac{\bar{E}}{\nu - i\omega_0}, \quad (13a)$$

and

$$v_{0y} = \frac{-(e/m)E_0\omega_c}{(\omega_0^2-\omega_c^2)}. \quad (13b)$$

By defining the effective third-order nonlinear polarization in terms of the effective third-order optical susceptibility, $P_e^{(3)} = \varepsilon_0\chi_e^{(3)}|E_0|^2|E_{ap}|$, and utilizing the expression given in Eq. (12), the complex effective third-order susceptibility $\chi_e^{(3)}$ of the semiconductor medium corresponding to the modulated (sideband) frequencies, $\omega_\pm$, can be derived. This susceptibility characterizes the strength of the nonlinear coupling between the interacting optical, plasma, and acoustical modes and governs the efficiency of the modulational amplification process. Accordingly, the resulting expression for $\chi_e^{(3)}$ is given by

$$\chi_e^{(3)} = \frac{2(e/m)^2\beta^2k^4v_{ap}\omega_p^2(\Omega^2-\nu^2)(X_2)^{-1}}{\varepsilon_0\rho\Omega_{ad}^2(\omega_0^2-\omega_c^2)^2}. \quad (14)$$

Equation (14) further shows that the effective third-order optical susceptibility, $\chi_e^{(3)}$, is generally a complex quantity. Consequently, it can be decomposed into its real and imaginary components, which respectively describe the nonlinear refractive (dispersive) response and the nonlinear gain or absorption characteristics of the semiconductor medium. Accordingly, the complex susceptibility may be expressed as:

$$[\chi_e^{(3)}] = [\chi_e^{(3)}]_r + i[\chi_e^{(3)}]_i, \quad (15a)$$

where

$$[\chi_e^{(3)}]_r = \frac{2(e/m)^2\beta^2k^4v_{ap}\omega_p^2(\Omega^2-\nu^2)(\omega_{ap}^2-k^2v_{ap}^2)(X_2)^{-1}}{\varepsilon_0\rho[(\omega_{ap}^2-kv_{ap}^2)^2+4\Gamma_{ap}^2\omega_{ap}^2)](\omega_0^2-\omega_c^2)^2}, \quad (15b)$$

and

$$[\chi_e^{(3)}]_i = \frac{-4(e/m)^2\beta^2k^4v_{ap}\omega_p^2(\Omega^2-\nu^2)\omega_{ap}\Gamma_{ap}(X_2)^{-1}}{\varepsilon_0\rho[(\omega_{ap}^2-kv_{ap}^2)^2+4\Gamma_{ap}^2\omega_{ap}^2)](\omega_0^2-\omega_c^2)^2}. \quad (15c)$$

Here, the subscripts "*r*" and "*i*" associated with the effective third-order susceptibility denote its real and imaginary components, respectively. The present analysis has been carried out in the heavily doped regime, where the free-carrier concentration is sufficiently high, while the semiconductor remains within the validity range of the adopted hydrodynamic approximation.

Equations (15b) and (15c) describe the steady-state nonlinear optical response of the semiconductor medium in the presence of a transverse external magnetic field and govern the propagation of the modulated electromagnetic wave. It is evident from these expressions that the effective nonlinear susceptibility is strongly influenced by both the equilibrium carrier concentration n_0, through the electron-plasma frequency ω_p, and the externally applied d.c. magnetic field B_0, through the electron-cyclotron frequency ω_c. These two parameters provide convenient means for controlling and tuning the nonlinear optical response of the semiconductor.

Furthermore, the real part of the effective third-order susceptibility, $[\chi_e^{(3)}]_r$, determines the nonlinear dispersive properties of the medium, whereas the imaginary part, $[\chi_e^{(3)}]_i$, governs the steady-state growth (or gain) of the modulated wave. In the present investigation, emphasis is placed exclusively on the modulational amplification of acoustical phonons in weakly piezoelectric magnetoactive III–V semiconductor crystals. A detailed analysis of the dispersive characteristics associated with the real part of the nonlinear susceptibility is beyond the scope of this work and will be presented in a future publication.

From Eq. (15b), it is evident that the real part of the effective third-order susceptibility, $[\chi_e^{(3)}]_r$, may assume either positive or negative values, depending on the sign of the factor $(\Omega^2-\nu^2)$, i.e., on the relative magnitudes of the electron-plasma frequency (ω_p) and the electron-cyclotron frequency (ν).

A positive value of $[\chi_e^{(3)}]_r$ corresponds to positive nonlinear dispersion of the modulated wave. Under this condition, the nonlinear refractive index decreases with increasing radial distance from the beam axis. According to Snell's law, the phase velocity of the modulated wave therefore increases away from the beam center, causing the optical beam to converge toward the axis. This behavior results in the well-known phenomenon of self-focusing.

Conversely, when $[\chi_e^{(3)}]_r$ becomes negative, the medium exhibits negative nonlinear dispersion. In this case, the nonlinear refractive index increases with radial distance from the beam axis, causing the propagating beam to diverge. Consequently, the optical field undergoes self-defocusing, leading to a reduction in beam confinement.

In the special case of a dispersionless acoustical phonon (AP) mode, i.e., when $\omega_{ap} \approx kv_{ap}$, the dispersion characteristics become anomalous, and $[\chi_e^{(3)}]_r$ approaches zero. This indicates that the induced nonlinear current density no longer contributes to the modification of the refractive index. Therefore, for a non-dispersive AP mode, the semiconductor medium does not exhibit any nonlinear refractive-index change arising from the induced current density.

To investigate the possibility of modulational amplification in the proposed semiconductor system, we employ the well-established relation reported in Ref. [33]. This relation provides the criterion for evaluating the amplification characteristics of the modulated wave in terms of the effective nonlinear response of the semiconductor medium and serves as the basis for determining the growth rate of the modulational instability. Accordingly, the required expression is given by

$$\alpha_e = \frac{k}{2\varepsilon_1}[\chi_e^{(3)}]_i |E_0|^2 , \tag{16}$$

where α_e denotes the effective nonlinear absorption coefficient of the semiconductor medium. Modulational amplification of the generated signal can occur only when the value of α_e, obtained from Eq. (16), is negative, indicating that the medium exhibits nonlinear optical gain rather than absorption.

By combining Eqs. (15c) and (16), it follows that the imaginary part of the effective third-order susceptibility, $[\chi_e^{(3)}]_i$, must also be negative in order to achieve a positive growth rate of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave. Accordingly, the conditions required for obtaining a positive modulational growth rate can be classified into the following two cases:

Case (i): When $\nu > \Omega$ and $\omega_0 < \omega_c$.

Case (ii): When $\nu < \Omega$ and $\omega_0 > \omega_c$.

For Case (i), it can be inferred that the presence of electron–electron collisions, even in the presence of an externally applied magnetic field, is not an essential requirement for the onset of modulational instability. Under this condition, the modulated signal wave with $\Omega < \nu$ can still experience amplification provided that the applied magnetic field is sufficiently strong such that the corresponding electron-cyclotron frequency exceeds the pump-wave frequency, i.e., $\omega_c > \omega_0$. Therefore, the application of an appropriate external magnetic field alone is capable of creating favorable conditions for modulational growth through the plasmon–acoustical phonon interaction.

For Case (ii), when $\Omega > \nu$, modulational amplification can be achieved by applying an external magnetic field such that the corresponding electron-cyclotron frequency remains lower than the pump-wave frequency, i.e., $\omega_c < \omega_0$. This indicates that both electron–electron collisions and the externally applied magnetic field play an important role in determining the conditions for the onset and evolution of modulational instability.

However, when $\omega_c > \omega_0$, the process may be significantly influenced by cyclotron absorption, which can suppress or dominate the instability mechanism. Moreover, satisfying the corresponding operating condition becomes experimentally challenging. Therefore, in the present investigation, the analysis has been confined to the second case, corresponding to $\nu > \Omega(= \omega_0 - \bar{\omega}_p)$, since this operating regime is more practical and can be readily realized under normal experimental conditions.

2.2 Threshold pump field for the onset of plasmon–acoustical phonon interaction based modulational amplification

As is well established in nonlinear wave theory, modulational amplification can occur only when the pump-field amplitude exceeds a certain threshold value. Below this critical pump intensity, the nonlinear coupling is insufficient to overcome the inherent losses in the medium, and consequently no net amplification of the modulated wave takes place. The threshold condition for the onset of modulational amplification is therefore obtained by setting $[\chi_e^{(3)}]_i = 0$, corresponding to the transition from attenuation to amplification. Applying this condition yields the following expression for the threshold pump-field amplitude:

$$\left|E_{0,th}\right|_{pl-ap} = \frac{\Omega(\omega_0^2 - \omega_c^2)}{(e/m)k\omega_0^2} . \tag{17}$$

Equation (17) indicates that the onset of modulational instability requires a finite threshold pump amplitude, even in the absence of damping mechanisms. In other words, the nonlinear interaction between the pump wave and the coupled plasmon–acoustical phonon modes becomes effective only when the pump-field amplitude exceeds a critical value.

The expression for the threshold pump amplitude, $(E_{0,th})_{pl-ap}$, corresponding to the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification, exhibits a complex dependence on the physical parameters of the semiconductor system. It is observed that $(E_{0,th})_{pl-ap}$ is essentially independent of the piezoelectric coefficient β, whereas it is strongly influenced by the wave number *k*, the equilibrium carrier concentration n_0 through the electron-plasma frequency ω_p (via the parameter Ω, and the externally applied d.c. magnetic field B_0 through the corresponding electron-cyclotron frequency ω_c. These parameters therefore provide effective means for controlling the threshold condition for modulational amplification in weakly piezoelectric magnetoactive III–V semiconductor crystals.

2.3 Growth rate of plasmon–acoustical phonon interaction based modulationally amplified wave

For pump-field amplitudes significantly exceeding the threshold value, the nonlinear interaction becomes sufficiently strong to produce sustained amplification of the modulated wave. Under these conditions, the growth rate of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave can be derived by combining Eqs. (15c) and (16). The resulting expression describes the exponential growth of the modulated signal and quantifies the efficiency of the nonlinear energy transfer from the pump wave to the coupled plasmon–phonon mode. Accordingly, the growth rate is given by

$$(g)_{pl-ap} = \frac{-(e/m)^2\beta^2 k^5 v_{ap}\omega_p^2(\Omega^2 - \nu^2)\omega_{ap}\Gamma_{ap}(X_2)^{-1}|E_0|^2}{\varepsilon_0\varepsilon_1\rho[(\omega_{ap}^2 - kv_{ap}^2)^2 + 4\Gamma_{ap}^2\omega_{ap}^2)](\omega_0^2 - \omega_c^2)^2} \tag{18}$$

A detailed analysis of the steady-state growth factor indicates that significant modulational amplification of the signal wave, with a growth rate of the order of 10^4–$10^6\,\mathrm{m}^{-1}$, can be achieved only under the condition of anomalous dispersion, i.e., when $\omega_{ap} \to kv_{ap}$. This demonstrates that anomalous dispersion provides the most favorable regime for efficient nonlinear energy transfer and amplification.

The theoretical formulation further shows that the externally applied magnetic field B_0, represented through the electron-cyclotron frequency ω_c, plays a crucial role in enhancing the growth rate of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave. By appropriately tuning the magnetic field strength, the modulational gain can be substantially increased.

It is also evident from the analysis that the growth rate of the modulated wave is independent of the sideband (signal) frequency $\omega_{\pm}$, while it is primarily governed by the pump-wave frequency ω_0 and the acoustical phonon frequency ω_{ap}. This theoretical prediction is consistent with the available experimental observations reported in the literature [41].

Furthermore, the growth rate is strongly influenced by several key physical parameters, including the wave number *k*, the equilibrium carrier concentration n_0, through the electron-plasma frequency ω_p (via the parameter Ω, and the externally applied d.c. magnetic field B_0, through the corresponding electron-cyclotron frequency ω_c. These parameters provide effective control over the efficiency of the modulational amplification process in weakly piezoelectric magnetoactive III–V semiconductor crystals.

3. Results and discussion

To demonstrate the applicability and validity of the proposed theoretical model, a weakly piezoelectric n-type III–V semiconductor crystal maintained at 77 K (liquid nitrogen temperature) has been selected as the host medium. The crystal is assumed to be illuminated by a pulsed CO_2 laser operating at a wavelength of 10.6 μm. At this temperature, the semiconductor exhibits a very low absorption coefficient in the vicinity of 10 μm, allowing the contribution from interband (band-to-band) electronic transitions to be safely neglected. Consequently, the nonlinear optical response is predominantly governed by free-carrier dynamics, making the adopted hydrodynamic model well suited for investigating plasmon–acoustical phonon interaction-induced modulational amplification. The material and laser parameters employed in the numerical computations are taken from the literature [42] and are listed below:

$m = 0.014m_0$; where m_0 is the free electron mass,
$\rho = 5.8\times10^3\,\mathrm{kg\,m^{-3}}$, $\beta = 0.054\,\mathrm{Cm^{-2}}$, $\Gamma_{ap} = 2\times10^{10}\ \mathrm{s^{-1}}$,
$\omega_{ap} = 2\times10^{11}\ \mathrm{s^{-1}}$, $v_{ap} = 4\times10^3\ \mathrm{ms^{-1}}$, $\nu = 4\times10^{11}\,\mathrm{s^{-1}}$,
$\varepsilon_1 = 15.8$.

The numerical values adopted in the present study correspond to a typical n-type InSb crystal and have been chosen to provide realistic quantitative estimates of the proposed modulational amplification mechanism. Nevertheless, the theoretical formulation developed in the preceding sections is sufficiently general and is not restricted to InSb alone. With appropriate modification of the relevant material parameters, such as the effective mass, dielectric constant, piezoelectric coefficient, and carrier concentration, the present model can be readily extended to investigate plasmon–acoustical phonon interaction-induced modulational amplification in other III–V semiconductor materials, including GaAs, GaSb, InAs, and related compound semiconductors.

3.1 Threshold characteristics for the onset of plasmon–acoustical phonon interaction based modulational amplification

Using the material parameters corresponding to n-type InSb listed in the previous section, the dependence of the threshold pump amplitude, $(E_{0,th})_{pl-ap}$, required for the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification, has been investigated numerically using Eq. (17). Particular attention has been given to the influence of key physical parameters, namely the wave number *k*, the externally applied magnetic field B_0, represented through the electron-cyclotron frequency ω_c, and the doping concentration n_0, characterized by the electron-plasma frequency ω_p. The computed results illustrating the variation of the threshold pump amplitude with these parameters are presented in Figs. 1–3.

Figure 1 illustrates the variation of the threshold pump amplitude, $(E_{0,th})_{pl-ap}$, required for the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification as a function of the wave number *k* for $\omega_c, \omega_p \approx \omega_0$. It is evident that the threshold pump amplitude is relatively high at smaller values of the wave number, indicating that a stronger pump field is required to initiate the modulational amplification process in this regime. As the wave number increases, $(E_{0,th})_{pl-ap}$ decreases in a nearly parabolic manner, implying that the nonlinear interaction becomes progressively more efficient at larger wave numbers.

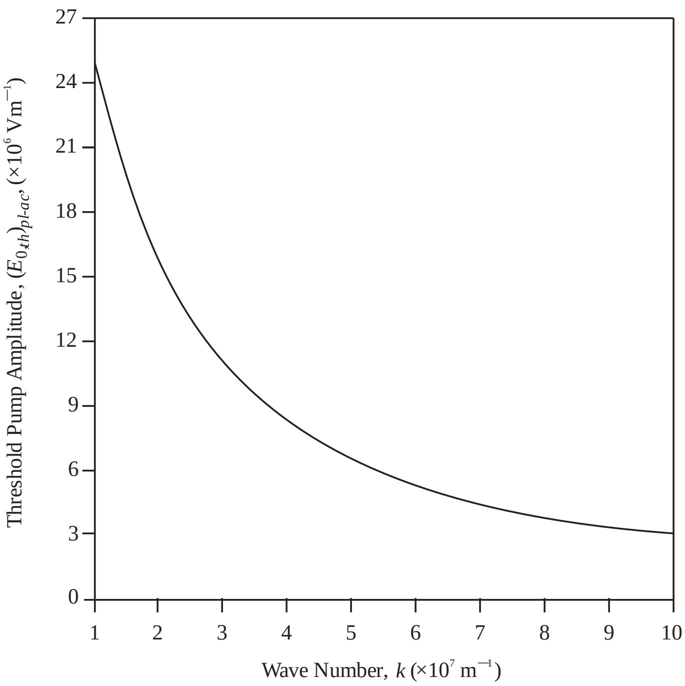

Figure 1: Variation of threshold pump amplitude $(E_{0,th})_{pl-ap}$ for the onset of plasmon–acoustical phonon interaction based modulational amplification with wave number k at $\omega_c, \omega_p \approx \omega_0$.

This behavior can be explained from Eq. (17), which predicts an inverse dependence of the threshold pump amplitude on the wave number through the corresponding analytical expression. Consequently, increasing the wave number enhances the coupling strength between the plasmon and acoustical phonon modes, thereby reducing the minimum pump-field amplitude required to trigger modulational amplification.

Figure 2 depicts the variation of the threshold pump amplitude, $(E_{0,th})_{pl-ap}$, for the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification as a function of the externally applied magnetic field B_0, (via ω_c/ω_0), for $\omega_p \approx \omega_0$ and $\omega_{ap} \approx kv_{ap}$. The results indicate that $(E_{0,th})_{pl-ap}$ remains almost unaffected by the magnetic field in the low-field regime, where $\omega_c < \omega_0$.

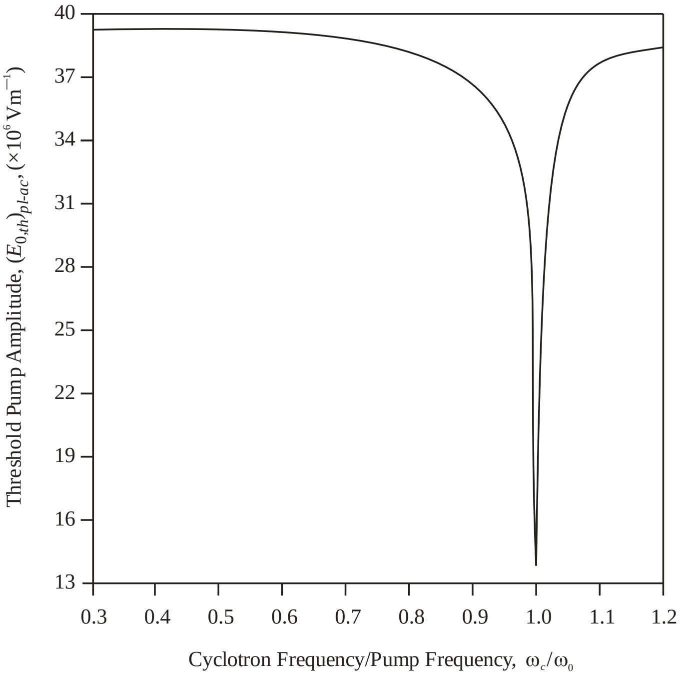

Figure 2: Variation of threshold pump amplitude $(E_{0,th})_{pl-ap}$ for the onset of plasmon–acoustical phonon interaction based modulational amplification with externally applied magnetic field B_0 (via ω_c/ω_0) at $\omega_p \approx \omega_0$ and $\omega_{ap} \approx kv_{ap}$.

As the magnetic field strength increases, the threshold pump amplitude decreases rapidly, indicating an enhancement in the nonlinear coupling between the interacting modes. A pronounced minimum is observed near $\omega_c \sim \omega_0$, where the threshold pump amplitude attains its lowest value ($\sim 1.4\times10^7$ Vm^{-1}}) for an n-type InSb crystal irradiated by a 10.6 µm CO_2 laser of frequency 1.78×10^{14} s^{-1}, corresponding to an external magnetic field of $B_0 = 14.2\,\mathrm{T}$. This minimum arises from the dependence $(E_{0,th})_{pl-ap} \propto (\omega_0^2 - \omega_c^2)$, which leads to an optimum balance between the plasma and cyclotron responses of the free carriers.

Beyond this optimum magnetic field, a further increase in B_0 causes the threshold pump amplitude to rise sharply. At sufficiently high magnetic fields, where $\omega_c > \omega_0$, $(E_{0,th})_{pl-ap}$ again becomes nearly insensitive to the magnetic field, indicating that the influence of magnetic tuning gradually saturates in the high-field regime.

Figure 3 illustrates the dependence of the threshold pump amplitude, $(E_{0,th})_{pl-ap}$, for the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification on the doping concentration n_0 (via ω_p/ω_0), for $\omega_c \approx \omega_0$ and $\omega_{ap} \approx kv_{ap}$. The numerical results show that the threshold pump amplitude decreases with increasing doping concentration, indicating that heavily doped semiconductor crystals require comparatively lower pump-field amplitudes to initiate the modulational amplification process.

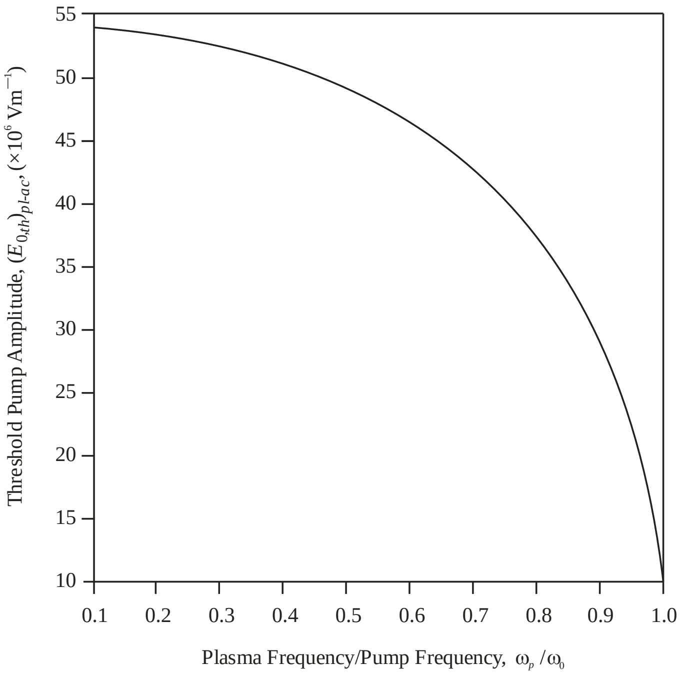

Figure 3: Variation of threshold pump amplitude $(E_{0,th})_{pl-ap}$ for the onset of plasmon–acoustical phonon interaction based modulational amplification with doping concentration n_0 (via ω_p/ω_0) at $\omega_c \approx \omega_0$ and $\omega_{ap} \approx kv_{ap}$.

At relatively low doping concentrations ($\omega_p < \omega_0$), the reduction in $(E_{0,th})_{pl-ap}$ is gradual. However, as the carrier concentration increases, the threshold pump amplitude decreases more rapidly owing to the enhancement of the free-carrier plasma response. A distinct minimum is observed near $\omega_p \sim \omega_0$, where the threshold pump amplitude reaches its lowest value ($\sim 10^7\,\mathrm{Vm}^{-1}$) for an n-type InSb crystal irradiated by a 10.6 µm CO_2 laser of frequency 1.78×10^{14} s^{-1}, corresponding to a doping concentration of 2.2×10^{24} m^{-3}. The occurrence of this minimum may be attributed to the fact that $(E_{0,th})_{pl-ap} \propto \Omega[=(\omega_0^2 - \bar{\omega}_p^2)^{1/2}]$, as predicted by Eq. (17), which yields an optimum carrier concentration for efficient nonlinear coupling.

3.2 Growth rate characteristics of plasmon–acoustical phonon interaction based modulationally amplified wave

Using the material parameters corresponding to n-type InSb listed in the previous section, the dependence of the growth rate, $(g)_{pl-ap}$, of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave has been numerically investigated using Eq. (18). The analysis has been carried out for pump-field amplitudes well above the threshold value to ensure that the modulational amplification process operates in the stable gain regime. The influence of several key parameters, namely the wave number *k*, the externally applied magnetic field B_0, represented through the electron-cyclotron frequency ω_c, the doping concentration n_0, characterized by the electron-plasma frequency ω_p, and the pump-field amplitude E_0, has been systematically examined. The corresponding numerical results illustrating the variation of the growth rate with these parameters are presented in Figs. 4–7.

Figure 4 presents the variation of the growth rate, $(g)_{pl-ap}$, of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave as a function of the wave number *k* for $\omega_c, \omega_p \approx \omega_0$ and a pump-field amplitude $E_0 = 2.5\times10^8$ Vm^{-1}. According to Eq. (18), the growth rate exhibits the characteristic dependence on the wave number, i.e., $(g)_{pl-ap} \propto [ak^2(b|E_0|^2 - ak^2)]^{1/2}$, which is consistent with the theoretical prediction reported by Drake et al. [43].

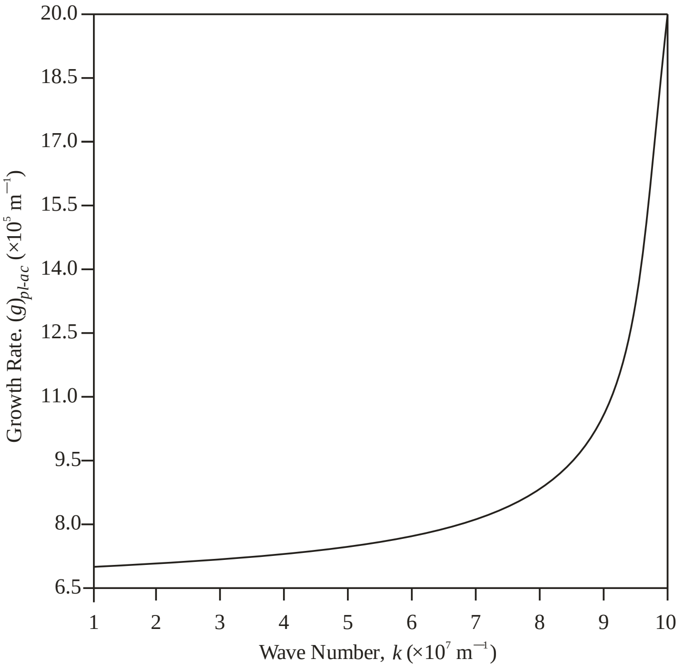

Figure 4: Variation of growth rate $(g)_{pl-ap}$ of plasmon–acoustical phonon interaction based modulationally amplified wave with wave number k at $\omega_c, \omega_p \approx \omega_0$ and $E_0 = 2.5\times10^8$ Vm^{-1}.

It is observed that for smaller values of the wave number k ($\le 6\times10^7$ m^{-1}), the growth rate increases gradually and approximately linearly with increasing wave number. However, with further increase in the wave number, the growth rate exhibits a more rapid enhancement and follows an approximately quadratic dependence on *k*. This behavior arises due to the increasing efficiency of the nonlinear coupling between the carrier-density perturbations and the acoustical phonon mode at higher wave-vector values.

The obtained variation of the temporal growth rate with wave number is therefore in agreement with the expected theoretical dependence, confirming the validity of the present model for describing PL–AP interaction-induced modulational amplification in weakly piezoelectric magnetoactive III–V semiconductor crystals.

Figure 5 illustrates the variation of the growth rate, $(g)_{pl-ap}$, of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave with the externally applied magnetic field B_0, (via ω_c/ω_0), for $\omega_p \approx \omega_0$, $\omega_{ap} \approx kv_{ap}$, and a pump-field amplitude $E_0 = 2.5\times10^8$ Vm^{-1}.

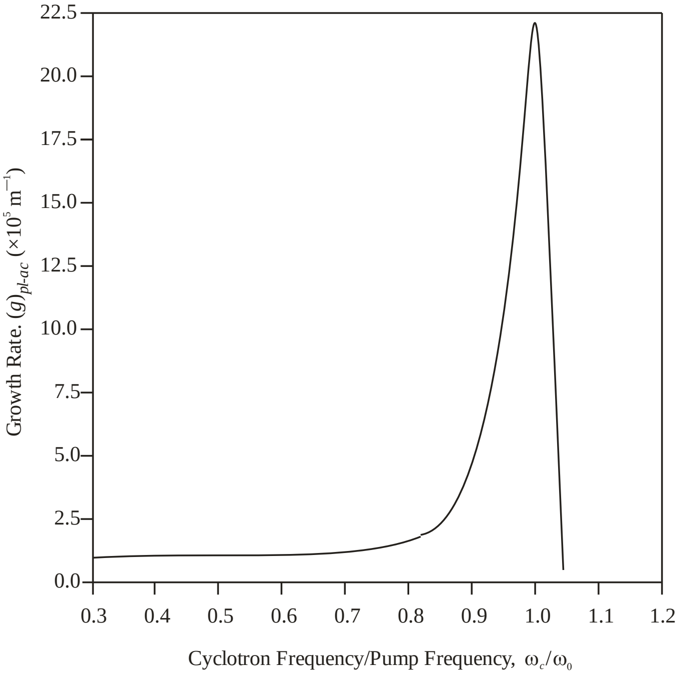

Figure 5: Variation of growth rate $(g)_{pl-ap}$ of plasmon–acoustical phonon interaction based modulationally amplified wave with externally applied magnetic field B_0 (via ω_c/ω_0) at $\omega_p \approx \omega_0$, $\omega_{ap} \approx kv_{ap}$ and $E_0 = 2.5\times10^8$ Vm^{-1}.

It is observed that the growth rate remains very small and almost independent of the applied magnetic field in the low-field region up to $\omega_c = 0.7\omega_0$. With a further increase in magnetic field strength, the growth rate increases rapidly due to the enhanced coupling between the carrier dynamics and the acoustical phonon mode. The growth rate reaches a maximum value near $\omega_c \approx \omega_0$ and subsequently decreases sharply, attaining a minimum around $\omega_c = 1.05\omega_c$.

The occurrence of the maximum growth rate can be attributed to the optimum interaction dependence $(g)_{pl-ap} \propto (\omega_0^2 - \omega_c^2)^{-2}$, as evident from Eq. (18). Beyond this optimum value, the rapid reduction in the growth rate is associated with the increasing contribution of the magnetic-field-dependent term appearing in the denominator of Eq. (18), which weakens the effective nonlinear gain.

The analysis further reveals that significant amplification cannot be obtained beyond $\omega_c = 1.05\omega_c$, as in this high magnetic-field regime the cyclotron absorption process becomes dominant and suppresses the modulational instability. The disappearance of the growth rate at and above this critical magnetic field defines a favorable operational range for magnetic-field-controlled amplification. This useful tunability of the gain characteristics with the applied magnetic field suggests potential applications in the development of magnetically controlled optical switching and optoelectronic modulation devices.

Figure 6 represents the variation of the growth rate, $(g)_{pl-ap}$, of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave with the doping concentration n_0, (via ω_p/ω_0), for $\omega_c \approx \omega_0$, $\omega_{ap} \approx kv_{ap}$, and a pump-field amplitude $E_0 = 2.5\times10^8$ Vm-1.

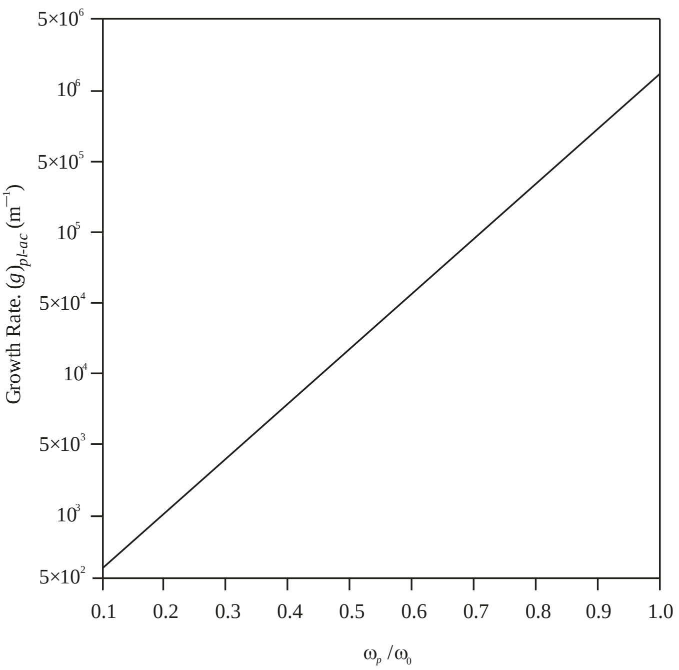

Figure 6: Variation of growth rate $(g)_{pl-ap}$ of plasmon–acoustical phonon interaction based modulationally amplified wave with doping concentration n_0 (via ω_p/ω_0) at $\omega_c \approx \omega_0$, $\omega_{ap} \approx kv_{ap}$ and $E_0 = 2.5\times10^8$ Vm-1.

It is observed that the growth rate remains very small ($\approx 10^2$ m-1) at low doping concentrations ($\omega_p \approx 0.1\omega_c$, corresponding $n_0 = 10^{22}$ m-3). As the carrier concentration increases, the growth rate increases almost linearly, indicating that enhanced free-carrier density strengthens the coupling between the plasmon mode and the acoustical phonon mode. At higher doping levels ($\omega_p \sim \omega_0$, corresponding $n_0 = 2.2\times10^{24}$ m-3), the growth rate becomes considerably larger ($\approx 10^6$ m-1), demonstrating the strong dependence of modulational amplification on the equilibrium electron density of the semiconductor medium.

The observed variation of the growth rate with carrier concentration is consistent with the results reported by Salimullah and Singh [44], who investigated the interaction of an extraordinary electromagnetic mode subjected to perturbations parallel to the magnetic field. The enhancement of the growth rate with increasing doping concentration can be attributed to the increase in the electron-plasma frequency, which improves the nonlinear coupling strength and facilitates more efficient energy transfer from the pump wave to the modulated modes.

Therefore, a higher modulational gain can be achieved by increasing the n-type doping concentration of the semiconductor crystal. However, the doping level must remain below the critical limit where the electron-plasma frequency exceeds the pump-wave frequency ($\omega_p > \omega_0$). Beyond this limit, the semiconductor behaves as an overdense plasma, resulting in reflection of the incident electromagnetic pump wave and preventing effective propagation through the medium. Hence, heavily doped III–V semiconductor crystals provide suitable platforms for achieving enhanced modulational instability and nonlinear optical amplification.

The preceding analysis has been carried out for III–V semiconductor materials, particularly n-type InSb, under conditions of relatively high doping levels approaching the critical carrier concentration. The critical density corresponds to the situation in which the associated electron-plasma frequency becomes comparable to the frequency of the incident pump wave.

Such highly doped regimes are practically significant for III–V compound semiconductors, as carrier concentrations of this order can be achieved through advanced doping techniques and are widely relevant for controlling their electronic and optical properties [45]. Several researchers have extensively employed heavily doped III–V semiconductor crystals to investigate their transport, optical, and nonlinear characteristics under different excitation conditions [46, 47]. Therefore, the parameter range considered in the present study represents a realistic and experimentally accessible regime for exploring plasmon–acoustical phonon interaction-based modulational amplification.

Figure 7 depicts the variation of the growth rate, $(g)_{pl-ap}$, of the plasmon–acoustical phonon (PL–AP) interaction-induced modulationally amplified wave with the pump-field amplitude E_0 for $\omega_c, \omega_p \approx \omega_0$ and $\omega_a \approx kv_a$. It is observed that for relatively lower pump-field amplitudes $E_0 \le 3\times10^8$ Vm-1, the growth rate increases almost linearly with increasing pump amplitude. This behavior indicates that the nonlinear coupling strength between the pump wave and the PL–AP modes increases proportionally with the applied optical excitation.

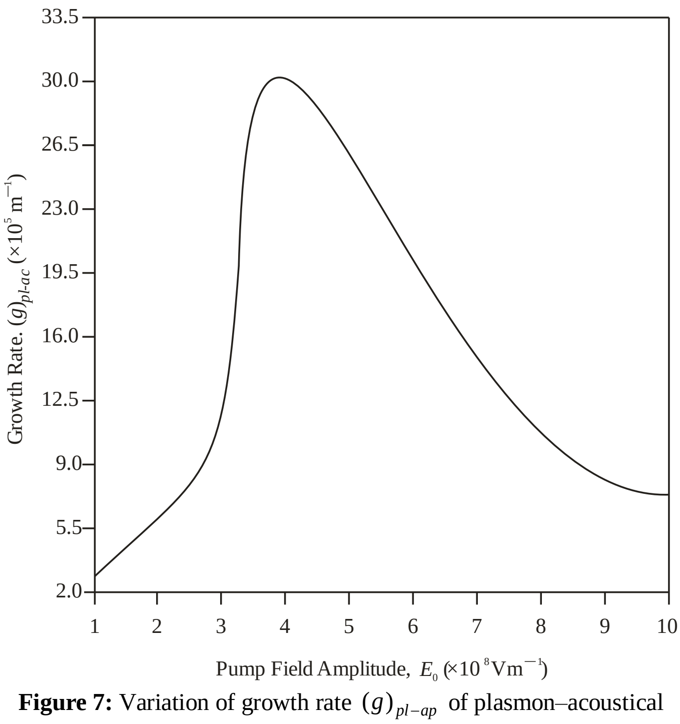

Figure 7: Variation of growth rate $(g)_{pl-ap}$ of plasmon–acoustical phonon interaction based modulationally amplified wave with pump field amplitude E_0 at $\omega_c, \omega_p \approx \omega_0$ and $\omega_{ap} \approx kv_{ap}$.

With further increase in the pump-field amplitude, the growth rate rises more rapidly and reaches a maximum value at an optimum pump amplitude of approximately $E_0 = 4\times10^8$ Vm-1. The occurrence of this peak corresponds to the condition where the energy transfer from the pump wave to the modulated plasmon–acoustical phonon mode becomes most efficient. Beyond this optimum value, the growth rate decreases gradually and exhibits an approximately linear reduction with further increase in pump-field intensity.

This decline may be attributed to the reduction in the effective nonlinear gain under excessively strong pumping conditions due to the modified carrier response and saturation effects. The obtained results demonstrate that the modulational amplification process can be effectively controlled by adjusting the pump-field amplitude. Therefore, an appropriate selection of the pump intensity enables the attainment of maximum growth rate of the PL–AP interaction-induced modulationally amplified wave in weakly piezoelectric magnetoactive III–V semiconductor crystals.

An approximate estimation of the third-order nonlinear optical susceptibility, $\chi^{(3)}$, of the semiconductor crystal can also be obtained within the framework of the present theoretical model. It should be noted that, in the proposed formulation, the contribution of virtual electronic transitions has been completely neglected. Instead, the origin of the induced nonlinear polarization has been attributed solely to the nonlinear current density generated by the collective motion of free charge carriers.

Therefore, the calculated value of the effective third-order susceptibility represents the nonlinear optical response arising predominantly from the free-carrier-induced piezoelectric Kerr mechanism associated with the plasmon–acoustical phonon interaction. This approach provides an estimation of the nonlinear susceptibility relevant to modulational amplification processes in weakly piezoelectric magnetoactive III–V semiconductor crystals.

Using Eq. (15) and considering appropriate values of the relevant physical parameters of the semiconductor medium, an approximate estimation of the effective third-order optical susceptibility, $\chi^{(3)}$, can be obtained. The calculated value represents the nonlinear optical response arising from the free-carrier-induced nonlinear current density, which acts as the primary source of the induced polarization in the present theoretical framework.

Since the contribution of virtual electronic transitions has been neglected, the estimated susceptibility corresponds mainly to the carrier-mediated nonlinear response associated with the plasmon–acoustical phonon (PL–AP) interaction. This provides a realistic measure of the nonlinear susceptibility responsible for modulational amplification in weakly piezoelectric magnetoactive III–V semiconductor crystals.

$$\chi^{(3)} \approx \frac{\omega_p^2 (e/m)^2 \varepsilon_1 \beta^2 k^4 v_{ap}}{\varepsilon_0 C \omega_{ap} \Gamma_{ap}^2 (\omega_0^2 - \omega_c^2)(\omega_0^2 - \bar{\omega}_p^2)} . \tag{19}$$

It has been experimentally observed that $\chi^{(3)} \approx 10^{-12}$ esu with $n_0 \approx 10^{22}$ m^{-3} [48], whereas from Eq. (19) we find that crystals with $k = 2.5\times10^7$ m^{-1} and $n_0 = 2\times10^{24}$ m^{-3}, yield $\chi^{(3)} \approx 10^{-7}$ esu. This close correspondence indicates that the proposed plasmon–acoustical phonon (PL–AP) interaction-based mechanism provides a realistic description of the nonlinear optical response in weakly piezoelectric III–V semiconductor crystals. It can therefore be inferred that the doping concentration of the semiconductor plays a significant role in enhancing the third-order optical susceptibility, primarily through the modification of the electron-plasma frequency and the associated free-carrier contribution to the nonlinear polarization.

Furthermore, the externally applied magnetic field, represented through the electron-cyclotron frequency ω_c, also provides an effective means of controlling and increasing the nonlinear susceptibility of the medium. Enhancement of $\chi^{(3)}$ through appropriate adjustment of carrier concentration and magnetic field strength enables the excitation of third-order nonlinear optical phenomena at comparatively lower pump-field amplitudes, thereby improving the feasibility of practical nonlinear optoelectronic applications.

4. Conclusions

In the present investigation, a comprehensive numerical analysis has been carried out to examine the threshold pump-field amplitude required for the initiation of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification and the corresponding growth rate of the modulated wave for pump amplitudes significantly above the threshold value in weakly piezoelectric magnetoactive doped III–V semiconductor crystals. The obtained results provide valuable insights into the influence of key controlling parameters, including the wave number, carrier concentration, and externally applied magnetic field, on the nonlinear amplification process.

Based on the theoretical formulation and numerical analysis, the following conclusions can be drawn:

1. The hydrodynamic model of semiconductor plasma has been effectively employed to investigate the influence of various controlling parameters, including the externally applied magnetostatic field and carrier concentration (doping level), on the threshold pump amplitude and the growth rate of modulated waves (for pump amplitudes well above the threshold value) in weakly piezoelectric magnetoactive doped III–V semiconductor crystals irradiated by slightly off-resonant, moderate-power pulsed laser sources.
2. The threshold pump amplitude required for the onset of plasmon–acoustical phonon (PL–AP) interaction-induced modulational amplification decreases with increasing wave number, indicating enhanced nonlinear coupling at higher spatial frequencies.
3. The application of an external transverse d.c. magnetic field, particularly under conditions where the electron-cyclotron frequency becomes comparable to the pump-wave frequency, significantly lowers the threshold pump amplitude required to initiate PL–AP interaction-based modulational amplification.
4. Higher doping concentrations, corresponding to electron-plasma frequencies approaching the pump-wave frequency, substantially reduce the threshold pump amplitude by strengthening the free-carrier-mediated nonlinear interaction.
5. The growth rate of the PL–AP interaction-induced modulationally amplified wave increases with increasing wave number, demonstrating enhanced energy transfer between the coupled plasmon and acoustical phonon modes.
6. The presence of an appropriate external transverse d.c. magnetic field effectively enhances the growth rate of the PL–AP interaction-based modulational amplification process through improved carrier-wave coupling.

7. The growth rate of the modulated wave exhibits a strong dependence on the doping concentration of the semiconductor crystal. It increases approximately linearly with increasing carrier concentration due to the enhanced contribution of free carriers to the nonlinear response.
8. Increasing the pump-field amplitude results in a corresponding enhancement of the growth rate, reaching an optimum maximum value at a pump amplitude of approximately $4\times10^{8}\,\mathrm{Vm^{-1}}$. Beyond this value, the growth rate decreases gradually due to the modified nonlinear response of the medium.

From the above analysis, it can be concluded that heavily doped III–V semiconductor crystals, particularly under optimized magnetic-field and pump-field conditions, provide highly suitable platforms for realizing modulational instability and plasmon–acoustical phonon interaction-based nonlinear optical amplification processes. The ability to control the amplification characteristics through external parameters highlights their potential for advanced optoelectronic devices, optical modulators, and nonlinear photonic applications.

Acknowledgements

The authors express their sincere gratitude to the Editor-in-Chief for their meticulous review, which significantly enhanced the quality of this work.

Authors' contributions

The authors read and approved the final manuscript.

Conflicts of interest

The authors declares no conflict of interest.

Funding

This research received no external funding.

Data availability

No new data were created.

References

[1] E. Agrell, M. Karlsson, A.R. Chraplyvy, D.J. Richardson, P.M. Krummrich, P. Winzer, K. Roberts, J.K. Fischer, S.J. Savory, B.J. Eggleton, M. Secondini, F.R. Kschischang, A. Lord, J. Prat, I. Tomkos, J.E. Bowers, S. Srinivasan, M. Brandt-Pearce, N. Gisin, Roadmap of optical communications, *J. Opt.* **18** (2016) 063002.

[2] H. Kaushal, G. Kaddoum, Underwater optical wireless communication, *IEEE Access* **4** (2016) 1518–1547.

[3] T. Xu, N.A. Shevchenko, D. Lavery, D. Semrau, G. Liga, A. Alvarado, R.I. Killey, P. Bayvel, Modulation format dependence of digital nonlinearity compensation performance in optical fibre communication systems, *Opt. Express* **25** (2017) 3311–3326.

[4] S.J. Sweeney, J. Mukherjee, Optoelectronic Devices and Materials, in: S. Kasap, P. Capper (Eds.), *Springer Handbook of Electronic and Photonic Materials*, Springer, Cham (2017).

[5] J. Cheng, C. Wang, X. Zou, L. Liao, Recent advances in optoelectronic devices based on 2D materials and their heterostructures, *Adv. Opt. Mater.* **6** (2018) 1800441.

[6] H. Shu, Z. Su, L. Huang, Z. Wu, X. Wang, Z. Zhang, Z. Zhou, Significantly high modulation efficiency of compact graphene modulator based on silicon waveguide, *Sci. Rep.* **8** (2018) 991.

[7] J. Lu, H. Liu, Y. Yu, Z. Ni, X. Zhang, Modulation of THz radiation via enhanced Dirac plasmon–dual phonon interaction, *Appl. Phys. Lett.* **115** (2019) 251109.

[8] Z.T. Ma, Z.X. Geng, Z.Y. Fan, J. Liu, H.D. Chen, Modulators for terahertz communication: The current state of the art, *Research* **2019** (2019) 6482975.

[9] P.J. Winzer, High-spectral efficiency optical modulation formats, *J. Lightwave Technol.* **30** (2012) 3824–3835.

[10] F. Copie, S. Randoux, P. Suret, The physics of the one-dimensional nonlinear Schrödinger equation in fiber optics: Rogue waves, modulation instability and self-focusing phenomena, *Rev. Phys.* **4** (2019) 100037.

[11] T.B. Benjamin, The disintegration of wave trains on deep water. Part 1. Theory, *J. Fluid Mech.* **27** (1967) 417–430.

[12] T.B. Benjamin, Instability of periodic wave trains in nonlinear dispersive systems, *Proc. R. Soc. Lond. A* **299** (1967) 59–76.

[13] G.P. Agrawal, *Nonlinear Fiber Optics*, 2nd edn., Academic Press, San Diego, CA (1995).

[14] H.C. Yuen, B.M. Lake, Instabilities of waves on deep water, *Annu. Rev. Fluid Mech.* **12** (1980) 303–334.

[15] Pravesh, S. Dahiya, D. Singh, et al., Quantum effects on modulational amplification in ion-implanted semiconductor magnetoplasmas, *Pramana – J. Phys.* **97** (2023) 58.

[16] E.F. Schubert, *Doping in III–V Semiconductors*, Cambridge University Press, Cambridge (2005).

[17] J. Sjakste, K. Tanimura, G. Barbarino, L. Perfetti, Hot electron relaxation dynamics in semiconductors: Assessing the strength of the electron–phonon coupling from the theoretical and experimental viewpoints, *J. Phys.: Condens. Matter* **30** (2018) 353001.

[18] E. Garmire, Resonant optical nonlinearities in semiconductors, *IEEE J. Sel. Top. Quantum Electron.* **6** (2000) 1094–1110.

[19] M. Singh, D. Joseph, S. Duhan, Nonlinear optical parameters of magnetoactive semiconductor plasmas, *Int. J. Mod. Phys. B* **22** (2008) 3877–3887.

[20] S.S. Li, High Speed III–V Semiconductor Devices, in: S.S. Li (Ed.), *Semiconductor Physical Electronics*, Springer, New York (2006).

[21] G. Packeiser, III/V semiconductors: Substrate materials and epitaxy, *Adv. Mater.* **2** (1990) 464–471.

[22] S. Mokkapati, C. Jagadish, III–V compound semiconductors for optoelectronic devices, *Mater. Today* **12** (2009) 22–32.

[23] J. Heyman, P.A. Crowell, T. Mueller, K. Unterrainer, Terahertz emission from GaAs and InAs in a magnetic field, *Phys. Rev. B* **64** (2001) 085202.

[24] M.B. Johnston, A. Corchia, A. Dowd, E.H. Linfield, Magnetic-field-induced enhancement of terahertz emission from III–V semiconductor surfaces, *Physica E* **13** (2002) 896–899.

[25] I. Nevinskas, F. Kadlec, C. Kadlec, R. Butkutė, A. Krotkus, Terahertz pulse emission from epitaxial n-InAs in a magnetic field, *J. Phys. D: Appl. Phys.* **52** (2019) 365301.

[26] R. Müller, Amplitude and frequency modulation of light pulses in semiconductor Fabry–Perot amplifiers, *Phys. Status Solidi B* **150** (1988) 587–592.

[27] L.N. Binh, *Optical Modulation: Advanced Techniques and Applications in Transmission Systems and Networks*, CRC Press, Taylor & Francis Group, Boca Raton (2018).

[28] J. Lekavich, Basics of acousto-optic devices, *Lasers Appl.* (1986) 59–64.

[29] Z. Feng, F. Yang, X. Zhang, D. Chen, F. Wei, N. Cheng, Y. Sun, Y. Gui, H. Cai, Ultra-low noise optical injection locking amplifier with AOM-based coherent detection scheme, *Sci. Rep.* **8** (2018) 13135.

[30] M. Hase, M. Katsuragawa, A.M. Constantinescu, H. Petek, Coherent phonon-induced optical modulation in semiconductors at terahertz frequencies, *New J. Phys.* **15** (2013) 055018.

[31] F. Iikawa, A.H. Mínguez, M. Ramsteiner, P.V. Santos, Optical phonon modulation in semiconductors by surface acoustic waves, *Phys. Rev. B* **93** (2016) 195212.

[32] P.K. Chakraborty, B.N. Mondal, B.K. Chaudhari, Laser-induced modulation of optical band-gap parameters in III–V semiconductors from density-of-states calculations, *Pramana – J. Phys.* **92** (2019) 85.
[33] A. Neogi, Acousto-optic modulation in diffusive semiconductors, *J. Appl. Phys.* **77** (1995) 327–333.
[34] S. Ghosh, M.P. Rishi, Acousto-optic modulation in magnetized diffusive semiconductors, *Eur. Phys. J. D* **19** (2002) 223–230.
[35] S. Ghosh, M.P. Rishi, Acousto-optic amplitude modulation and demodulation of electromagnetic waves in magnetized diffusive semiconductor plasma, *Eur. Phys. J. D* **20** (2002) 275–279.
[36] N. Nimje, S. Dubey, S.K. Ghosh, Diffusion-induced modulational instability in magnetized semiconductor plasmas: Effects of carrier heating, *Eur. Phys. J. D* **59** (2010) 223–231.
[37] M. Först, T. Dekorsy, Coherent phonons in bulk and low-dimensional semiconductors, in: S. De Silvestri (Ed.), *Coherent Vibrational Dynamics*, CRC Press, Boca Raton (2008) pp. 129–172.
[38] D.L. Spears, Brillouin scattering study of propagating acoustoelectric domains in *n*-GaAs, *Phys. Rev. B* **2** (1970) 1931–1951.
[39] J.J. Wynne, N. Bloembergen, Measurement of the lowest-order nonlinear susceptibility in III–V semiconductors by second-harmonic generation with a CO_2 laser, *Phys. Rev. B* **2** (1970) 4306–4308.
[40] P.K. Kaw, Parametric excitation of ultrasonic waves in piezoelectric semiconductors, *J. Appl. Phys.* **44** (1973) 1497–1498.
[41] A. Yariv, *Optical Electronics*, 3rd edn., Holt, Rinehart and Winston (Saunders College Publishing), New York (1984) pp. 395–401.
[42] S. Bhan, H.P. Singh, V. Kumar, Acoustical-phonon and polaron mode-induced optical parametric amplification in transversely magnetized III–V semiconductors, *Mod. Phys. Lett. B* **33** (2019) 1950271.
[43] J.F. Drake, P.K. Kaw, Y.C. Lee, G. Schmidt, C.S. Liu, M.N. Rosenbluth, Parametric instabilities of electromagnetic waves in plasmas, *Phys. Fluids* **17** (1974) 778–785.
[44] M. Salimullah, T. Singh, Filamentation and modulational instabilities of laser radiation in magnetoactive piezoelectric semiconductors, *J. Phys. Chem. Solids* **43** (1982) 1087–1092.
[45] K. Seeger, *Semiconductor Physics*, Springer-Verlag, New York (1973).
[46] P.K. Sen, N. Apte, S. Guha, Raman instability in *n*-type piezoelectric semiconductors, *Phys. Rev. B* **22** (1980) 6340–6346.
[47] P. Kumari, B.S. Sharma, M. Singh, Hot carrier effects on Brillouin susceptibilities of semiconductor magnetoplasmas, *Pramana – J. Phys.* **96** (2022) 49.
[48] C. Flytzanis, Third-order optical susceptibilities in IV–IV and III–V semiconductors, *Phys. Lett. A* **31** (1970) 273–274.